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Quantum Mutual Entropy for a Multilevel Atom Interacting with a Cavity Field

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We derive an explicit formula for the quantum mutual entropy as a measure of the total correlations in a multi-level atom interacting with a cavity field. We describe its theoretical basis and discuss its practical relevance. The effect of the number of levels involved on the quantum mutual entropy is demonstrated via examples of three-, four- and five-level atom. Numerical calculations under current experimental conditions are performed and it is found that the number of levels present changes the general features of the correlations dramatically.

KEY WORDS: multi-level systems; mutual entropy.

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1. OVERVIEW

Great progress has recently been made in quantum information theory (Nielsen and Chuang, 2000). Also, entropy becomes a fundamental quantity to describe not only uncertainty or chaos of a system but also information carried by the system (Ohya and Petz, 1993). Compared to the long history of the theoretical understanding of entropy and entanglement of atom-field systems extending over many decades (Fichtner and Ohya, 2001), intensive experimental investigations started only recently involving different systems (Eisert and Plenio, 1999). To identify the fundamentally inequivalent ways quantum systems can be entangled is a major goal of quantum information theory (Eisert and Plenio, 1999). It might be thought that there is nothing new to be said about bipartite entanglement if the shared state is pure, but in a recent paper (Rekdal, 2004) it has been shown that exact coherence of the atom is in general never regained for a two-level model

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with a general initial pure quantum state of the radiation field. Also, it has been shown that the purification of the atomic state is actually independent of the nature of the initial pure state of the radiation field.

From the viewpoint of the Phoenix and Knight (1988, 1991a,b) entropy formalism, the quantum field entropy and entanglement of a coherent field interacting with a three-level systems have been investigated (Abdel-Aty, 2000, 2003). However, the method used in those papers cannot be applied when the system is taken to be initially in a mixed state. A method using quantum mutual entropy to measure the entanglement in the time development of the two-level system model has been adopted in Furuichi (1999). The question of how mixed a two-level system and a field mode may be such that free entanglement arises in the course of the time evolution according to a Jaynes–Cummings type interaction has been considered (Bennett *et al.*, 1996; Bose *et al.*, 2001a,b,c; Hill and Wootters, 1997; Nielsen and Kempe, 2001).

It is important to point out that further insights into the dynamics of the multi-level systems may be helpful in developing quantum information theory (Vedral and Plenio, 1998). Recently, there is much interest in multi-level quantum systems to represent information (Abdel-Aty *et al.*, 2002; Furuichi and Abdel-Aty, 2001; Furuichi and Ohya, 1999; Scheel *et al.*, 2003). It was demonstrated that key distributions based on multi-level quantum systems are more secure against eavesdropping than those based on two-level systems (Furuichi and Ohya, 1999). Key distribution protocols based on entangled three-level systems were also proposed (Bose *et al.*, 2001a,b,c). The security of these protocols is related to the violation of the Bell inequality. The multi-level system provides in this context a much smaller level of noise (Cerf *et al.*, 2002; Song *et al.*, 2002; Zheng, 1998). Rydberg atoms crossing superconductive cavities are an almost ideal system to generate entangled states, and to perform small scale quantum information processing (Bourennane *et al.*, 2001). In this context entanglement generation of multi-level quantum systems was also reported (Acin *et al.*, 2002; Bruss and Macchiavello, 2002; Durt *et al.*, 2003; Kaszlikowski *et al.*, 2000; Vedral and Plenio, 1998).

Our motivation is to generalize the quantum mutual entropy, usually employed in the two-level system, to the multi-level system interacting with a cavity field. This is because the quantum mutual entropy can be thought of as the original correlation measure of mixed (rather than simply pure) input states. Using an appropriate representation and without using the diagonal approximation, an explicit expression for quantum mutual entropy when the system starts from a mixed state is derived. Although various special aspects of the quantum mutual entropy have been investigated previously, the general features of the dynamics, when a multi-level system is considered, have not been treated before and the present paper therefore fills a gap in the literature. The physical situation which we shall refer to, belongs to the experimental domains of cavity quantum electrodynamics.

The plan of the remainder of the paper as follows. In Section 2, we go through a more rigorous set of definitions leading up to the exact solution of the multi-level system and give exact expression for the unitary operator U_t involved. In Section 3, we consider one of the intermediate definitions namely the quantum mutual entropy and develop several results related to this quantity. These include a more convenient expression that automatically takes into account an arbitrarily number of atomic levels, where the atom is initially in the mixed state, depending on both the value of the total correlations and on the measurements required for its definition. In Section 4, we show how difficult it is to derive rigorously the quantum mutual entropy involving more than three-levels. The final part of this article is devoted to some important developments of the quantum mutual entropy and we close the paper with a list of open questions.

2. THE MULTI-LEVEL SYSTEM

We start by devoting this section to a brief discussion on the multi-level atom (Abdel-Hafez *et al.*, 1987, 1989) being it the model describing the interaction between a single multi-level atom and a quantized cavity field. To set the stage, we first begin by describing the multilevel-atom model. Therefore, the physical system on which we focus is an m-level. The atom interacts with a high Q-cavity which sustain a number of modes of the field with frequencies Ω_j, $j = 1, 2, \ldots, m-1$. We denote by $\hat{a}_j$ and $\hat{a}_j^\dagger$ the annihilation and creation operators for the field mode j, and ω_j is the frequency associated with the level of the atom. We assume that the mode i affects the transition between the upper atomic level and the level $(i+1)$. Therefore in the rotating wave approximation we can cast the Hamiltonian of the system in the form (Abdel-Hafez *et al.*, 1989) ($\hbar = 1$)

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{1}$$

where the Hamiltonian for the interacting system $\hat{H}_0$ is given by

$$\hat{H}_0 = \sum_{j=1}^{m-1} \Omega_j \hat{a}_j^\dagger \hat{a}_j + \sum_{i=1,2,m\ldots} \omega_i |i\rangle\langle i|.$$

The interaction Hamiltonian between the atomic system and the cavity field is given by

$$\hat{H}_1 = \sum_{j=1}^{m-1} (\lambda_j (\hat{S}_{1,j+1} \hat{a}_j + h.c.)).$$

The transition in the m-level atom is characterized by the coupling λ_i. The operator $\hat{S}_{ii}$ describes the atomic population of level $|i\rangle_A$ with energy ω_i, $(i = 1, 2, \ldots, m)$

and the operator $\hat{S}_{ij} = |i\rangle\langle j|, (i \neq j)$ describes the transition from level $|i\rangle_A$ to level $|j\rangle_A$.

We have applied the rotating wave approximation discarding the rapidly oscillating terms and selecting the terms that oscillate with minimum frequency (Vogel and de Matos Filho, 1995). The resulting effective Hamiltonian may be written as

$$\hat{H}_0 = (\omega_1 - \Delta)I + \sum_{j=1}^{m-1} \Omega_j(\hat{a}_j^\dagger \hat{a}_j - \hat{S}_{j+1,j+1}), \tag{2}$$

$$\hat{H}_1 = \Delta \hat{S}_{11} + \sum_{j=1}^{m-1} \lambda_i(\hat{S}_{1,j+1}\hat{a}_i + \hat{S}_{j+1,1}\hat{a}_i^\dagger) \tag{3}$$

We have used $\sum_{i=1}^{m} = I$. Here we assume that the detuning parameter Δ is given by

$$\Delta = \omega_1 - \omega_{j+1} - \Omega_j, \quad j = 1, 2, \ldots\ldots m-1.$$

It can be shown that $\hat{H}_0$ and $\hat{H}_1$ are constants of motion,

$$[\hat{H}_0, \hat{H}_1] = [\hat{H}, \hat{H}_0] = 0. \tag{4}$$

We assume that, before entering the cavity, the atom is prepared in a mixed state. Mixed states arise when there is some ignorance with respect to the system, so that consideration has to be given to the possibility that the system is in any one of several possible states, S_{ii}, each with some probability, γ_i, of being realized. To this end, the initial state of the atom can be written in the following form

$$\rho = (\gamma \hat{S}_{11} + \gamma_2 \hat{S}_{22} + \gamma_3 \hat{S}_{33} + \cdots + \gamma_m \hat{S}_{mm}) \in S_A, \tag{5}$$

where $\gamma_i \geq 0$, and $\sum_{i=1}^{m} \gamma_i = 1$. In terms of quantum information processes, an understanding of mixed states is essential, as it is almost inevitable that the ideal pure states will interact with the environment at some stage.

Also we suppose that the initial state of the field is given by

$$|\bar{\omega}_1\rangle = \left(\sum_{n_1, n_2, \ldots = 0}^{\infty} b_{n_1} b_{n_2} \ldots b_{n_{m-1}} |n_1, n_2 \ldots n_{m-1}\rangle \right) \in S_F, \tag{6}$$

where $b_{n_i} = \langle \varpi | n_i \rangle$, $b_{n_i}^2$ being the probability distribution of photon number for the initial state. The continuous map $\mathcal{E}_t^*$ describing the time evolution between the atom and the field is defined by the unitary operator generated by $\hat{H}$ such that

$$\mathcal{E}_t^* \; : \; S_A \to S_A \otimes S_F,$$

$$\mathcal{E}_t^* \rho = \hat{U}_t(\rho \otimes \varpi)\hat{U}_t^*,$$

$$\hat{U}_t \equiv \exp\left(-\frac{i}{\hbar}\int \hat{H}(t)dt\right). \tag{7}$$

where $\varphi = |\varphi_1\rangle\langle\varphi_1|$. Bearing these facts in mind we find that the evolution operator $\hat{U}_t$ takes the next from

$$\hat{U}_t \equiv \exp\left(-\left(\omega_1 - \Delta\right)t\right)\left[\prod_{j=1}^{m-1}\exp\left(-i\Omega_j\widehat{N}_j t\right)\right]\exp\left(-i\int_0^t \hat{H}_t dt\right). \tag{8}$$

where $\widehat{N}_j = \hat{a}_j^\dagger\hat{a}_j - S_{j+1,j+1}$. The first two factors in Eq. (11) produce phases that will not affect the results that follow, while calculations of the third factor show that it takes the following compact matrix form

$$\exp\left(-i\hat{H}_1 t\right) = \exp\left(-\frac{i}{2}\Delta t\right)\begin{bmatrix}\widehat{U}_0 & \widehat{U}_1^* \\ \widehat{U}_1 & \widehat{U}_2\end{bmatrix}, \tag{9}$$

where $\widehat{U}_0$ is the single element matrix $\{\widehat{U}_1\}$ which takes the following form

$$\widehat{U}_{11} = \cos\hat{\mu}_n t - \frac{i\Delta}{2}\frac{\sin\hat{\mu}_n t}{\hat{\mu}_n}. \tag{10}$$

The matrix $\widehat{U}_1^*$ is the $1 \times (m-1)$ row matrix $\{\widehat{U}_{1k}\}$, where

$$\widehat{U}_{1k} = -i\frac{\sin\hat{\mu}_n t}{\hat{\mu}_n}\lambda_k\hat{a}_k, \quad k \in \{1, 2, 3, \ldots, m-1\} \tag{11}$$

and U_{BA} its Hermitian conjugate. Finally the matrix $\widehat{U}_2$ is the $(m-1) \times (m-1)$ square matrix $\{\widehat{U}_{ij}\}$ of which the elements can be written as

$$\widehat{U}_{ij} = \delta_{ij}\exp\left(-\frac{i}{2}\Delta t\right) - \lambda_i\hat{a}_i^\dagger v^{-1}\left(\cos\hat{\mu}_n t - \exp\left(-\frac{i}{2}\Delta t\right) + \frac{i\Delta}{2}\frac{\sin\hat{\mu}_n t}{\hat{\mu}_n}\right)\lambda_j\hat{a}_j, \tag{12}$$

with $i, j = 1, 2, \ldots, m-1$ and

$$\hat{\mu}_n = \left(\frac{\Delta^2}{4} + \sum_{i=1}^{m-1}\lambda_i^2\hat{a}_i\hat{a}_i^\dagger\right)^{\frac{1}{2}}, \quad v^{-1} = \sum_{i=1}^{m-1}\lambda_i^2\hat{a}_i\hat{a}_i^\dagger \tag{13}$$

Having obtained the explicit form of the unitary operator U_t, we are therefore able to discuss the total correlations of the system.

3. DERIVATION OF THE QUANTUM MUTUAL ENTROPY

The study of mutual entropy in classical system was extensively done after Shannon (Gelfand and Yaglom, 1959; Kolmogorov, 1963). In quantum systems, there have been several definitions of the mutual entropy for classical input and quantum output (Belavkin and Stratonovich, 1973; Holevo, 1973; Ingarden, 1976) and the fully quantum mechanical mutual entropy by means of the relative entropy has been investigated (Levitin, 1991; Ohya, 1983). On the other hand, quantifying the amount of entanglement between quantum systems is a recent pursuit that has attracted a diverse range of researchers (Abdel-Aty, 2000; Abdel-Aty and Obada, 2003; Abdel-Aty *et al.*, 2002; Bennett *et al.*, 1996; Bose *et al.*, 2001a,b,c; Furuichi and Abdel-Aty, 2001; Furuichi and Ohya, 1999; Hill and Wootters, 1997; Nielsen and Kempe, 2001; Phoenix and Knight, 1988, 1991a,b; Rekdal *et al.*, 2004; Scheel *et al.*, 2003; Vedral and Plenio, 1998). When we look at the entanglement of the mixed state as a whole, we can calculate the relative entropy of entanglement (Vedral, 2004a,b) and for quantifying the total correlations we use the quantum mutual information.

In this section, we will apply the results obtained previously to derive the quantum mutual entropy for a single multi-level atom interacting with a cavity field without using the diagonal approximation method adapted in Abdel-Aty *et al.* (2002) and Furuichi and Abdel-Aty (2001). With a certain unitary operator, the final state after the interaction between the atom and the field is given by

$$\begin{aligned}\mathcal{E}_t^*\rho &= U_t\,(\rho\otimes\varpi)\,U_t^* \\ &= \gamma_1 U_t|a\varpi\rangle\left\langle \varpi, a|U_t^* + \sum_{i=1}^{m-1}\gamma_i U_t|b_i,\varpi\right\rangle\langle\varpi, b_i|U_t^*. \end{aligned} \tag{14}$$

Therefore the von Neumann entropy of the total system is given by

$$\mathcal{S}(\varepsilon_t^*\rho) = -\sum_{i=1}^{m}\gamma_i\log\gamma_i. \tag{15}$$

Taking the partial trace over the atomic system, we obtain

$$\rho_t^F = tr_A\mathcal{E}_t^*\rho.$$

Then the von Neumann entropy for the reduced state $S(\rho_t^F)$ is computed by

$$S\left(\rho_t^F\right) = -\sum_{i=1}^{m^2}\lambda_i^F(t)\log\lambda_i^F(t), \tag{16}$$

where $\{\lambda_i^F(t)\}$ are the solutions of

$$\det[\rho(\hat{t}) - \lambda(\hat{t})N(\hat{t})] = 0, \tag{17}$$

where $\rho(\hat{t})$ and $N(\hat{t})$ are $m^2\times m^2$ matrices having the following elements

$$\begin{aligned}&[\rho(\hat{t})]_{ij} = \langle\psi_i(t)|\rho_t^F|\psi_j(t)\rangle, \quad (i, j = 1, 2, 3, \ldots m^2), \\ &[N(\hat{t})]_{ij} = \langle\psi_i(t)|\psi_j(t)\rangle, \quad (i, j = 1, 2, 3, \ldots m^2),\end{aligned} \tag{18}$$

and $|\psi_j(t)\rangle$ are the eigenfunctions of the following eigenvalue problem $\rho_t^F|\psi_i(t)\rangle = \lambda_i^F(t)|\psi_i(t)\rangle$.

On the other hand, the final state of the atomic system is given by taking the partial trace over the field system:

$$\rho_t^A = tr_F \mathcal{E}_t^* \rho.$$

Then the von Neumann entropy for the reduced state $S(\rho_t^A)$ is computed by

$$S\left(\rho_t^A\right) = -\sum_{i=1}^{m} \lambda_i^A(t) \log \lambda_i^A(t), \tag{19}$$

where $\lambda_i^A(t)$ can be calculated by obtaining the eigenvalues of the reduced atomic state. Using the above equations, the final expression for the quantum mutual entropy for the m-level system takes the following form

$$\begin{aligned} I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right) &\equiv tr\mathcal{E}_t^*\rho\left(\log \mathcal{E}_t^*\rho - \log\left(\rho_t^A \otimes \rho_t^F\right)\right) \\ &= \sum_{i=1}^{m} \gamma_i \log \gamma_i - \sum_{i=1}^{m^2} \lambda_i^F(t) \log \lambda_i^F(t) \\ &\quad - \sum_{i=1}^{m} \lambda_i^A(t) \log \lambda_i^A(t). \end{aligned} \tag{20}$$

It turns out to be rather easy to derive an analytic expression for the quantum mutual entropy for any given system, since with the help of Eq. (20) it is possible to study the quantum mutual entropy of any m-level system when the system starts from its mixed state.

This seems significant, and one then wonders whether the trend might continue with the general multi-atom (or ions) case. That is, whether one might be able to consider more than one atom and still be able to calculate the quantum mutual entropy. Also, one might wonder whether a similar effect could carry over to different field states. That would be very nice because one could contemplate different protocols for different initial states of the field. To go a step further towards a deterministic quantum mutual entropy, we note a peculiar effect in the present paper: we get more correlations with increasing m (number of levels). Indeed in the limit that $\gamma_i \sim 0, i > 1$, (i.e. $\gamma_1 \approx 1$), the quantum mutual entropy is only twice of the quantum field (atomic) entropy. In the general case i.e., $\gamma_1 \neq 1$), the final state does not necessarily become a pure state, so that we need to make use of $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ in the present model. Thus our initial setting enables us to discuss the variation of the quantum mutual entropy for different values of the parameter γ_i of the initial atomic system. A related model allowing an analytic treatment of the quantum mutual entropy as well as valuable insight, namely the two-level atom ($m = 2$) has been discussed in Abdel-Aty *et al.* (2002) and Furuichi and

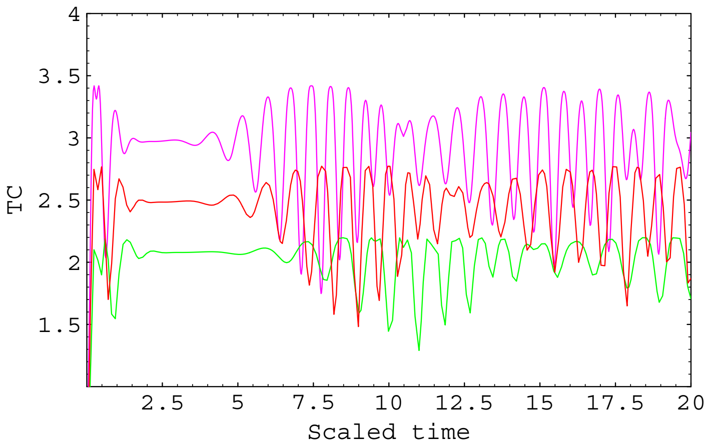

Fig. 1. The evolution of the quantum mutual entropy $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ as a function of the scaled time. The mean photon number $\bar{n} = 5$, and the detuning parameter Δ has zero value, where, from bottom to top depicts three-, four- and five-level atom, respectively.

Abdel-Aty (2001). An example of a truly mixed state for which the entanglement manipulations have been proven to be asymptotically reversible has been reported in Audenaert *et al.* (2003).

Here we focus on the time development of the quantum mutual entropy for some special cases such as three-, four- and five-level atoms. In Fig. 1, we plot the function $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ which describes the quantum mutual entropy in the case when the field is initially in a coherent state with a mean photon number $\bar{n} = 5$, and the mixed state parameters $\gamma_1 = 0.99$. In this case we see that, $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ oscillates around values nearly equals the maximum values $(2\ln(m))$. Let us remark that, in the pure state case, the von Neumann entropy is limited by $\ln(m)$, and then $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ reduces to $2\ln(m)$. From this figure we can say that the maximum value of $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ is increased as the number of levels is increased.

Nevertheless, the minimum values lie within the region between the two maximum values occurring in a similar way for different number of levels, such that with higher m the minimum values of the quantum mutual entropy occur at earlier times. In fact, for some higher values of m there were no persisting periods found to lie between the maximum and minimum values. These results strongly indicate that the higher number of levels give higher mutual entropy as well as more oscillations. Figure 2, indicates that when the mean photon number is increased further the minimum values of the quantum mutual entropy occur at later times. The reason why maximum value of the quantum mutual entropy is $2\ln(m)$ and not $\ln(m)$, is that, the quantum mutual information quantifies the total correlations,

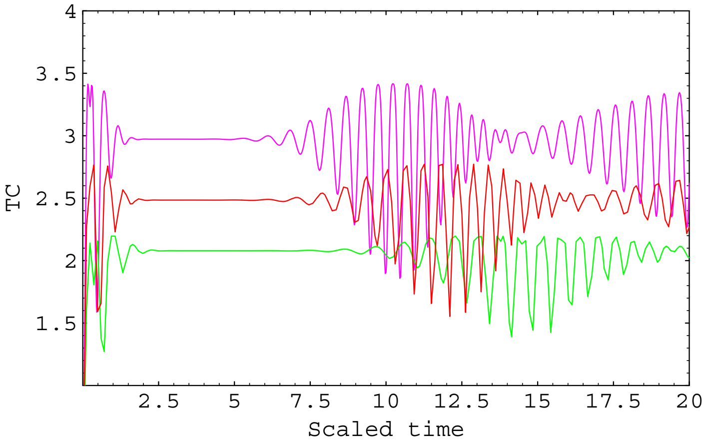

Fig. 2. The same as in Fig. 1 but now $\bar{n} = 10$.

both classical and quantum, where one $\ln(m)$ goes to the classical correlations and the other $\ln(m)$ goes to the quantum correlations (i.e. entanglement). In Fig. 3, we consider the quantum mutual entropy as a function of the scaled time with the field initially in a Fock state. The Fock state of the electromagnetic field is very difficult to produce in experiments. Nevertheless, these states are very important in quantum optics because of their intrinsic quantum nature. This case is quite

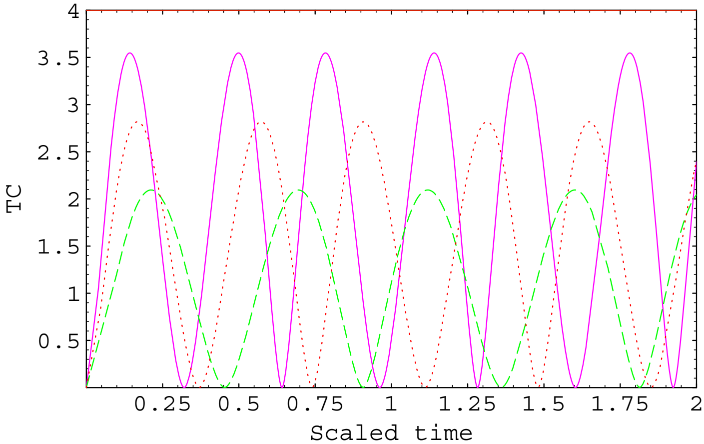

Fig. 3. The evolution of the quantum mutual entropy $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ as a function of the scaled time. In this figure we consider the Fock state with $n = 5$, where, from bottom to top depicts three-, four- and five-level atom, respectively.

interesting because the quantum mutual entropy function oscillates around the maximum and minimum values as time goes on. We have shown here a new phenomena where the periodic oscillations occur irrespective of number of atomic levels involved. This reflects the various influences of the initial states of the field. A slight change in n therefore, dramatically alters the quantum mutual entropy. It should be noted that for a special choice of the initial state setting, the situation becomes interesting where we find that a higher multi-level atom interacting with an initially coherent field exhibits superstructures instead of the usual first-order revivals.

It is worth mentioning that the dynamics of quantum multi-level systems has always been of interest, but has recently attracted even more attention because of application in quantum computing. Several systems have been suggested as physical realizations of quantum bits allowing for the needed control manipulations, and for some of them the first elementary steps have been demonstrated in experiments (Braunstein and Lo, 2001).

4. CONCLUSION

Summarizing, we have shown how to determine the maximum and minimum possible values of the quantum mutual entropy for multi-level atoms interacting with a cavity field. The forms of states that achieve these maximum and minimum values are the same as those for the case of the von Neumann entropy if we consider the pure state case. We have identified the relation between the quantum mutual entropy and von Neumann entropy. The general formula we have derived may carry over to any multi-level system. For the examples we have examined the quantum mutual entropy for three-, four- and five-level atoms, we have found that as the number of levels increases the maximum values of the quantum mutual entropy also increases, but these values are achieved for earlier times when the number of levels is increased accordingly.

An open and very interesting question is whether the quantum mutual entropy technique that we have described here can be transferred to other systems in which atomic and cavity decays are present. In those systems it may be possible to augment or simplify the definition of the quantum mutual entropy making its applications more accessible.

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